

# THE DIVISOR MATRIX, DIRICHLET SERIES AND $\mathrm{SL}(2, \mathbf{Z})$ , II

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ABSTRACT. We examine an elliptic curve constructed in an earlier paper from a certain representation of  $\mathrm{SL}(2, \mathbf{Z})$  on the space of convergent Dirichlet series. The curve is observed to be a modular curve for  $\Gamma^1(15)$  and a certain orbit of modular functions is thereby associated with the Riemann zeta function. Explicit descriptions are given of these functions and of the permutation action of  $\mathrm{SL}(2, \mathbf{Z})$  on them. One of the functions has a zero in the arc  $\{e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\}$ . The values of this particular function along paths formed from  $\mathrm{SL}(2, \mathbf{Z})$ -images of the arc are used to construct paths between zeros of the zeta function.

This paper is a continuation of [5]. Here we examine some consequences of Theorems 9.1 and 10.3.

If  $X, Y$  are indeterminates, set  $x = X + 1, y = Y/X$ . We check that

$$(x - 1)y^2 + xy - x(x - 1) = \frac{Y^2 + XY + Y - X^3 - X^2}{X}.$$

Since

$$(\zeta(s) - 1)\phi(s)^2 + \zeta(s)\phi(s) - \zeta(s)(\zeta(s) - 1) = 0, \quad \operatorname{Re}(s) \gg 0,$$

it follows that

$$(1) \quad Y^2 + XY + Y = X^3 + X^2,$$

where  $X = \zeta - 1, Y = \phi/\zeta$ . As the equation (1) is in long Weierstrass form [3, page 2], we compute that it has discriminant  $-15$ , conductor  $15$ , and  $j$ -invariant  $-1/15$ . Thus, (1) and the Taniyama-Weil Conjecture lead to  $\Gamma(15)$ . The curve is labeled 15A in [1] and 15A8 in J. Cremona's tables (<http://www.warwick.ac.uk/staff/J.E.Cremona/ftp/data/>). Following Shimura [4] and Fricke [2], if  $n \in \mathbf{N}$ , set

$$\Gamma(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}(2, \mathbf{Z}) \mid a \equiv d \equiv 1, b \equiv c \equiv 0 \pmod{n} \right\},$$

$$\Gamma^0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1) \mid b \equiv 0 \pmod{n} \right\},$$

$$\Gamma^1(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma^0(n) \mid a \equiv d \equiv 1 \pmod{n} \right\}.$$

If  $G$  is a subgroup of  $\Gamma(1)$  of finite index,  $X(G)$  denotes the function field of  $G$ .

Following Fricke [2], set

$$\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - q^n), \quad q = e^{2\pi iz}, \quad \eta_m(z) = \eta\left(\frac{z}{m}\right), \quad m \in \mathbf{N}.$$

It is a classical fact that  $X(\Gamma^1(15))$  is of genus 1 and has  $j$ -invariant  $-1/15$ . Therefore, there exist functions  $Z$  and  $\Phi$  of the upper half plane which generate  $X(\Gamma^1(15))$  as a field extension of  $\mathbf{C}$  and satisfy the equation

$$(Z(z) - 1)\Phi(z)^2 + Z(z)\Phi(z) - Z(z)(Z(z) - 1) = 0, \quad \forall z \in \mathbb{H}^*.$$

We now proceed to give an explicit construction of such functions  $Z$  and  $\Phi$ .

By Fricke [2],  $X(\Gamma^0(15)) = \mathbf{C}(\tau, \sigma)$ , where

$$\tau = (\eta_3\eta_5/\eta_1\eta_{15})^3$$

and

$$(2) \quad \sigma^2 = \tau^4 - 10\tau^3 - 13\tau^2 + 10\tau + 1 = (\tau - \alpha)(\tau - \alpha')(\tau - \beta)(\tau - \beta'),$$

where

$$\alpha = \frac{-1}{2} - \frac{\sqrt{5}}{2}, \quad \alpha' = \frac{-1}{2} + \frac{\sqrt{5}}{2}, \quad \beta = \frac{11}{2} - \frac{5\sqrt{5}}{2}, \quad \beta' = \frac{11}{2} + \frac{5\sqrt{5}}{2}.$$

Since  $-I$  fixes every element of  $X(G)$  for every subgroup  $G$  in  $\Gamma(1)$  of finite index, it follows that  $\langle -I, \Gamma^1(15) \rangle$  is the largest subgroup of  $\Gamma(1)$  which acts trivially on  $X(\Gamma^1(15))$ . Since  $\langle -I, \Gamma^1(15) \rangle \triangleleft \Gamma^0(15)$  and  $\Gamma^0(15)/\langle -I, \Gamma^1(15) \rangle$  is cyclic of order 4, it follows that  $X(\Gamma^1(15))$  is a Galois extension of  $X(\Gamma^0(15))$  and that  $\text{Gal}(X(\Gamma^1(15))/X(\Gamma^0(15))) \cong \mathbf{Z}/4\mathbf{Z}$ . Let

$$\lambda = \eta_1^{-3} \eta_3^6 \eta_5^{-3}.$$

Then  $\lambda$  is related to  $\tau$  and  $\sigma$  by the equation

$$\lambda^2 = \frac{c\sigma + d}{250\tau^4},$$

where  $c = \sum_{i=0}^4 f_i \tau^i$  and  $d = \sum_{i=0}^6 e_i \tau^i$  with

$$\begin{aligned} f_0 &= -1, f_1 = -19, f_2 = -104, f_3 = -125, f_4 = 125, \\ e_0 &= 1, e_1 = 24, e_2 = 180, e_3 = 374, e_4 = -396, e_5 = -750, e_6 = 125. \end{aligned}$$

Let  $H$  be the subgroup of index 2 in  $\Gamma^0(15)$  containing  $K = \langle -I, \Gamma^1(15) \rangle$ . Using the standard transformation formulae for  $\eta$  and generators for  $H$ , one sees that  $\lambda$  is fixed by  $H$ . Next, we let  $Z$  be a branch of the function defined by the equation

$$(3) \quad \left( \tau^2 \lambda + \frac{3^3 5^{-3/2} \tau^3}{\lambda} \right) (Z^2 - 3Z + 1) = \sqrt{\frac{5 + 2\sqrt{5}}{3}} (\tau - \beta)(\tau^2 + \gamma\tau + \delta)(Z^2 - Z + 1),$$

with  $\gamma = -\frac{1}{2} - \frac{21}{50}\sqrt{5}$  and  $\delta = -\frac{1}{10} - \frac{3}{50}\sqrt{5}$ . To make a definite choice of a germ of  $Z$  at  $i$ , we observe that  $\tau(i)$  and  $\lambda(i)$  are real, and pick  $Z$  so that  $Z(i)$  has positive imaginary part. It is straightforward to check that

$$(4) \quad B_1 \tau + B_0 = 0,$$

with

$$B_i = \sum_{j=0}^4 b_{ij} Z^j, \quad i = 0, 1,$$

$$\begin{aligned}
b_{00} = b_{04} = 1 \quad & b_{01} = b_{03} = -\frac{7}{2} - \frac{3}{2}\sqrt{5}, \quad b_{02} = 6 + 3\sqrt{5} \\
b_{10} = b_{14} = -\frac{1}{2} - \frac{\sqrt{5}}{2}, \quad & b_{11} = b_{13} = -2 + \sqrt{5}, \quad b_{12} = \frac{9}{2} - \frac{3}{2}\sqrt{5}.
\end{aligned}$$

Hence,

$$C(\sigma, \tau, Z) = \mathbf{C}(\sigma, Z).$$

From (2) and (3), we get

$$B_1^4 \sigma^2 = (B_0 + \alpha B_1)(B_0 + \alpha' B_1)(B_0 + \beta B_1)(B_0 + \beta' B_1).$$

We compute that

$$\begin{aligned}
B_0 + \alpha' B_1 &= -3\sqrt{5}Z(Z-1)^2, \\
B_0 + \alpha B_1 &= \left(\frac{5+\sqrt{5}}{2}\right)(Z^2 - Z + 1)^2, \\
B_0 + \beta' B_1 &= -(2 + \sqrt{5})(Z-1)^2(4Z^2 - 7Z + 4), \\
B_0 + \beta B_1 &= \left(\frac{9}{2} - \frac{3}{2}\sqrt{5}\right)(Z^2 + 3Z + 1)^2. \\
-3\sqrt{5} \left(\frac{5+\sqrt{5}}{2}\right) (-2 - \sqrt{5}) \left(\frac{9}{2} - \frac{3}{2}\sqrt{5}\right) &= 3^2 \cdot 5 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^2.
\end{aligned}$$

Set

$$\Psi = \frac{B_1^2 \sigma}{3\sqrt{5} \left(\frac{1+\sqrt{5}}{2}\right) (Z-1)^2 (Z^2 - Z + 1) (Z^2 + 3Z + 1)},$$

so that

$$(5) \quad \Psi^2 = 4Z^3 - 7Z^2 + 4Z,$$

and

$$X(\sigma, \tau, Z) = \mathbf{C}(\Psi, Z).$$

Set

$$(6) \quad \Phi = \frac{\Psi - Z}{2(Z-1)},$$

so that

$$X(\sigma, \tau, Z) = \mathbf{C}(\Phi, Z).$$

By (5) and (6),

$$(Z-1)\Phi^2 + Z\Phi - Z(Z-1) = 0.$$

We summarize these calculations in the following statement.

**Theorem II.1.**  $X(\Gamma^1(15))$  is generated as a field extension of  $\mathbf{C}$  by the functions  $Z$  and  $\Phi$  defined above, which satisfy

$$(Z(z) - 1)\Phi(z)^2 + Z(z)\Phi(z) - Z(z)(Z(z) - 1) = 0, \quad \forall z \in \mathbb{H}^*.$$

We compute that

$$|\Gamma(1) : \langle -I, \Gamma^1(15) \rangle| = 96,$$

and we construct an explicit set of coset representatives  $\{P_i \mid 1 \leq i \leq 96\}$ , so that

$$\Gamma(1) = \bigcup_{i=1}^{96} \langle -I, \Gamma^1(15) \rangle P_i.$$

These are given in the Appendix. We set

$$(7) \quad Z_i = Z^{P_i}, \quad \mathfrak{Z} = \{Z_i \mid 1 \leq i \leq 96\}.$$

The elements of  $\mathfrak{Z}$  are called *avatars* of  $\zeta$ .

We check that if  $z_0 \in \mathbb{H}^*$  and  $Z(z_0) = 0$ , then

$$\tau(z_0) = \frac{-1 + \sqrt{5}}{2}, \quad \sigma(z_0) = 0.$$

and from Fricke [2, page 450, equations (17) and (18)],

$$j = \frac{(\tau_5^2 + 10\tau_5 + 5)^3}{\tau_5}, \quad \tau_5 = (\eta_5/\eta_1)^6,$$

$$\tau_5 = \frac{\tau^4 - 9\tau^3 - 9\tau - 1 + (\tau^2 - 4\tau - 1)\sigma}{2\tau},$$

so

$$j(z_0) = 135 \left( \frac{1415 + 637\sqrt{5}}{2} \right) \approx 629.$$

Set

$$E = \{e^{i\theta} \mid \frac{\pi}{2} \leq \theta \leq \frac{2\pi}{3}\}.$$

Since  $E$  and  $[0, 1728]$  are in bijection via  $z \mapsto j(z)$ ,  $z \in E$ , it follows that there is a unique  $c \in E$  such that

$$j(c) = 135 \left( \frac{1415 + 637\sqrt{5}}{2} \right).$$

Since  $j(z) = j(z')$  if and only if  $z$  and  $z'$  are in the same  $\Gamma(1)$ -orbit, it follows that there is  $g_0 \in \Gamma(1)$  such that  $Z(g_0(c)) = 0$ . A straightforward calculation yields that

$$Z^{g_0} = Z_{41}.$$

Set

$$T = \bigcup_{g \in \Gamma(1)} g(E).$$

We check that if  $g \in \Gamma(1)$  and  $E \cap g(E) \neq \emptyset$ , then one of the following holds:

1.  $g = \pm I$  and  $E = g(E)$ ,
2.  $g = \pm R$  and  $E \cap g(E) = \{\omega\}$ ,  $\omega = e^{\frac{2\pi i}{3}}$ ,
3.  $g = \pm S$  and  $E \cap g(E) = \{i\}$ .

This implies that  $T$  is a tree, with vertex set

$$V(T) = \{g(i) \mid g \in \Gamma(1)\} \cup \{g(\omega) \mid g \in \Gamma(1)\},$$

and edge set

$$E(T) = \{g(E) \mid g \in \Gamma(1)\}.$$

In particular, if  $t_1, t_2 \in T$ , then there is a unique path from  $t_1$  to  $t_2$ .

It is natural to consider elements  $g \in \Gamma(1)$  such that  $Z_{41}^g = Z_{41}$  and then to consider

$$\Delta = \{(z, s) \in \mathbb{H}^* \times \mathbf{C} \mid Z_{41}(z) = \zeta(s)\},$$

the motivation being that  $Z_{41}$  is an avatar of  $\zeta$ .

We consider the directed path  $P_g$  in  $T$  from  $c$  to  $g(c)$ . Thus for each zero  $\rho$  of  $\zeta$ ,  $(c, \rho) \in \Delta$ . We then use analytic continuation to build a path  $Q_{g, \rho}$  in  $\mathbf{C}$  such that as we traverse  $P_g$  in  $\mathbb{H}^*$  we simultaneously traverse  $Q_{g, \rho}$  in  $\mathbf{C}$  such that if  $(z, s) \in \mathbb{H}^* \times \mathbf{C}$  and  $z, s$  are corresponding points, then  $(z, s) \in \Delta$ . An injudicious choice of  $g$  will have a point  $z$  on  $P_g$  such that  $z$  is a pole of  $Z_{41}$ , and our way is blocked. However, the element

$$A = RSRSR^{-1}SRSR^{-1}SRSR^{-1}SRSR = (RSR)^4 = \begin{pmatrix} 0 & -1 \\ 1 & 3 \end{pmatrix}^4$$

fixes  $Z_{41}$  and  $Z_{41}$  has no poles on  $P_A$ .

Let  $\rho_1, \rho_2, \dots$  be zeros of  $\zeta$  in the upper half plane, ordered so that  $\rho_m = \frac{1}{2} + i\gamma_m$ , with  $0 < \gamma_1 < \gamma_2 < \dots$ . Using the computer program SAGE (MAXIMA, Pari) [6], we found that  $\rho_{m+1} \in Q_{A, \rho_m}$  for  $1 \leq m \leq 300$ . That is,  $(A(c), \rho_{m+1}) \in \Delta$ , as  $Q_{A, \rho_m}$  starts at  $\rho_m$  and ends at  $\rho_{m+1}$  for  $1 \leq m \leq 300$ . We do not understand this phenomenon, but it is sufficiently arresting to be noted explicitly.

APPENDIX. COSET REPRESENTATIVES  $P_i$  OF  $\langle -I, \Gamma^1(15) \rangle$  IN  $\Gamma(1)$ 

$n$	$P_n$	$w_n$	$nR$	$nS$
1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	24	24
2	$\begin{pmatrix} 1 & 3 \\ -1 & -2 \end{pmatrix}$	$R^{-1}SRSR$	23	20
3	$\begin{pmatrix} -2 & 3 \\ 1 & -2 \end{pmatrix}$	$R^{-1}SRSR^{-1}S$	19	23
4	$\begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}S$	44	22
5	$\begin{pmatrix} -2 & -3 \\ 1 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR$	22	21
6	$\begin{pmatrix} -5 & -3 \\ 2 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SRSR^{-1}$	21	43
7	$\begin{pmatrix} -2 & 5 \\ 1 & -3 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}S$	36	18
8	$\begin{pmatrix} 2 & 1 \\ -1 & 0 \end{pmatrix}$	$R^{-1}SR^{-1}$	11	14
9	$\begin{pmatrix} -4 & 1 \\ 3 & -1 \end{pmatrix}$	$R^{-1}SRSRSRS$	55	17
10	$\begin{pmatrix} -1 & -2 \\ 1 & 1 \end{pmatrix}$	$R^{-1}SR$	8	16
11	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$	$R^{-1}S$	10	15
12	$\begin{pmatrix} 5 & -2 \\ -2 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SRSRS$	57	13
13	$\begin{pmatrix} 2 & 5 \\ -1 & -2 \end{pmatrix}$	$R^{-1}SR^{-1}SRSR$	6	12
14	$\begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}S$	5	8
15	$\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$	$R^{-1}$	1	11
16	$\begin{pmatrix} -2 & 1 \\ 1 & -1 \end{pmatrix}$	$R^{-1}SRS$	2	10
17	$\begin{pmatrix} -1 & -4 \\ 1 & 3 \end{pmatrix}$	$R^{-1}SRSRSR$	28	9
18	$\begin{pmatrix} -5 & -2 \\ 3 & 1 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}$	3	7
19	$\begin{pmatrix} 3 & 5 \\ -2 & -3 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR$	18	30
20	$\begin{pmatrix} 3 & -1 \\ -2 & 1 \end{pmatrix}$	$R^{-1}SRSRS$	17	2
21	$\begin{pmatrix} -3 & 2 \\ 1 & -1 \end{pmatrix}$	$R^{-1}SR^{-1}SRS$	13	5
22	$\begin{pmatrix} 3 & 1 \\ -1 & 0 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}$	14	4
23	$\begin{pmatrix} -3 & -2 \\ 2 & 1 \end{pmatrix}$	$R^{-1}SRSR^{-1}$	16	3
24	$\begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}$	$R$	15	1

$n$	$P_n$	$w_n$	$nR$	$nS$
25	$\begin{pmatrix} -4 & -15 \\ 3 & 11 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}SRSRSR$	48	48
26	$\begin{pmatrix} -4 & 3 \\ 1 & -1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SRS$	47	44
27	$\begin{pmatrix} -8 & -3 \\ 3 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SRSR^{-1}SR^{-1}$	43	47
28	$\begin{pmatrix} 4 & 3 \\ -3 & -2 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}$	20	46
29	$\begin{pmatrix} 7 & 3 \\ -5 & -2 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}SR^{-1}$	46	45
30	$\begin{pmatrix} 5 & -3 \\ -3 & 2 \end{pmatrix}$	$R^{-1}SRSR^{-1}SRS$	45	19
31	$\begin{pmatrix} 7 & 5 \\ -3 & -2 \end{pmatrix}$	$R^{-1}SR^{-1}SRSRSR^{-1}$	12	42
32	$\begin{pmatrix} -7 & 4 \\ 5 & -3 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}SRS$	35	38
33	$\begin{pmatrix} -7 & -2 \\ 4 & 1 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}$	7	65
34	$\begin{pmatrix} -4 & 7 \\ 1 & -2 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SRSR^{-1}S$	32	40
35	$\begin{pmatrix} 4 & 11 \\ -3 & -8 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}SRSR$	34	39
36	$\begin{pmatrix} 5 & 7 \\ -3 & -4 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR$	33	37
37	$\begin{pmatrix} 8 & 5 \\ -5 & -3 \end{pmatrix}$	$R^{-1}SRSR^{-1}SRSR^{-1}$	30	36
38	$\begin{pmatrix} -4 & -7 \\ 3 & 5 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}SR$	29	32
39	$\begin{pmatrix} 11 & -4 \\ -8 & 3 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}SRSR$	25	35
40	$\begin{pmatrix} -7 & -4 \\ 2 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SRSR^{-1}$	26	34
41	$\begin{pmatrix} 4 & 1 \\ -1 & 0 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}$	4	81
42	$\begin{pmatrix} 5 & 8 \\ -2 & -3 \end{pmatrix}$	$R^{-1}SR^{-1}SRSR^{-1}SR$	27	31
43	$\begin{pmatrix} -3 & 5 \\ 1 & -2 \end{pmatrix}$	$R^{-1}SR^{-1}SRSR^{-1}S$	42	6
44	$\begin{pmatrix} -3 & -4 \\ 1 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR$	41	26
45	$\begin{pmatrix} -3 & -8 \\ 2 & 5 \end{pmatrix}$	$R^{-1}SRSR^{-1}SRSR$	37	29
46	$\begin{pmatrix} 3 & -4 \\ -2 & 3 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}S$	38	28
47	$\begin{pmatrix} 3 & 7 \\ -1 & -2 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SRSR$	40	27
48	$\begin{pmatrix} 15 & 11 \\ -11 & -8 \end{pmatrix}$	$R^{-1}SRSRSR^{-1}SRSRSR^{-1}$	39	25



$n$	$P_n$	$w_n$	$nR$	$nS$
49	$\begin{pmatrix} -7 & -30 \\ 4 & 17 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SRSRSR$	72	72
50	$\begin{pmatrix} -2 & 9 \\ 1 & -5 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SR^{-1}S$	95	68
51	$\begin{pmatrix} -1 & -6 \\ 1 & 5 \end{pmatrix}$	$R^{-1}SRSRSRSR$	67	71
52	$\begin{pmatrix} 7 & 9 \\ -4 & -5 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SR$	68	70
53	$\begin{pmatrix} -4 & 9 \\ 3 & -7 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR^{-1}S$	94	69
54	$\begin{pmatrix} 5 & 9 \\ -4 & -7 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR$	69	67
55	$\begin{pmatrix} 1 & 5 \\ -1 & -4 \end{pmatrix}$	$R^{-1}SRSRSRSR$	60	66
56	$\begin{pmatrix} -16 & 7 \\ 9 & -4 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SRSR$	59	62
57	$\begin{pmatrix} -2 & -7 \\ 1 & 3 \end{pmatrix}$	$R^{-1}SR^{-1}SRSRSR$	31	89
58	$\begin{pmatrix} 7 & -1 \\ -6 & 1 \end{pmatrix}$	$R^{-1}SRSRSRSRSRSR$	56	88
59	$\begin{pmatrix} 7 & 23 \\ -4 & -13 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SRSRSR$	58	63
60	$\begin{pmatrix} -5 & -4 \\ 4 & 3 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}$	9	61
61	$\begin{pmatrix} -4 & 5 \\ 3 & -4 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}S$	54	60
62	$\begin{pmatrix} -7 & -16 \\ 4 & 9 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SRSR$	77	56
63	$\begin{pmatrix} 23 & -7 \\ -13 & 4 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SRSRSR$	49	59
64	$\begin{pmatrix} -11 & -2 \\ 6 & 1 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}$	50	82
65	$\begin{pmatrix} -2 & 7 \\ 1 & -4 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}S$	52	33
66	$\begin{pmatrix} 5 & -1 \\ -4 & 1 \end{pmatrix}$	$R^{-1}SRSRSRSR$	51	55
67	$\begin{pmatrix} 6 & 5 \\ -5 & -4 \end{pmatrix}$	$R^{-1}SRSRSRSRSR^{-1}$	66	54
68	$\begin{pmatrix} -9 & -2 \\ 5 & 1 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SR^{-1}$	65	50
69	$\begin{pmatrix} -9 & -4 \\ 7 & 3 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR^{-1}$	61	53
70	$\begin{pmatrix} 9 & -7 \\ -5 & 4 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SR$	62	52
71	$\begin{pmatrix} -6 & 1 \\ 5 & -1 \end{pmatrix}$	$R^{-1}SRSRSRSRSR$	88	51
72	$\begin{pmatrix} 30 & 23 \\ -17 & -13 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SRSRSRSR^{-1}$	63	49

$n$	$P_n$	$w_n$	$nR$	$nS$
73	$\begin{pmatrix} -13 & -30 \\ 10 & 23 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR^{-1}SR^{-1}SRSR$	96	96
74	$\begin{pmatrix} -7 & 9 \\ 3 & -4 \end{pmatrix}$	$R^{-1}SR^{-1}SRSRSRSR^{-1}S$	71	92
75	$\begin{pmatrix} 4 & 9 \\ -1 & -2 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SRSR$	91	95
76	$\begin{pmatrix} 2 & 9 \\ -1 & -4 \end{pmatrix}$	$R^{-1}SR^{-1}SRSRSRSR$	92	94
77	$\begin{pmatrix} 1 & -6 \\ 0 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}S$	70	93
78	$\begin{pmatrix} -5 & -6 \\ 1 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}SR$	93	91
79	$\begin{pmatrix} -4 & -5 \\ 1 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SR$	84	90
80	$\begin{pmatrix} -4 & 13 \\ 3 & -10 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR^{-1}SR^{-1}S$	83	86
81	$\begin{pmatrix} 1 & -4 \\ 0 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}S$	79	41
82	$\begin{pmatrix} -2 & 11 \\ 1 & -6 \end{pmatrix}$	$R^{-1}SRSR^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}S$	80	64
83	$\begin{pmatrix} 13 & 17 \\ -10 & -13 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR^{-1}SR^{-1}SR$	82	87
84	$\begin{pmatrix} 5 & 1 \\ -1 & 0 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}$	81	85
85	$\begin{pmatrix} 1 & -5 \\ 0 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}S$	78	84
86	$\begin{pmatrix} -13 & -4 \\ 10 & 3 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR^{-1}SR^{-1}$	53	80
87	$\begin{pmatrix} 17 & -13 \\ -13 & 10 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR^{-1}SR^{-1}SRS$	73	83
88	$\begin{pmatrix} 1 & 7 \\ -1 & -6 \end{pmatrix}$	$R^{-1}SRSRSRSRSRSR$	74	58
89	$\begin{pmatrix} -7 & 2 \\ 3 & -1 \end{pmatrix}$	$R^{-1}SR^{-1}SRSRSR$	76	57
90	$\begin{pmatrix} -5 & 4 \\ 1 & -1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SRS$	75	79
91	$\begin{pmatrix} -9 & -5 \\ 2 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SRSR^{-1}$	90	78
92	$\begin{pmatrix} -9 & -7 \\ 4 & 3 \end{pmatrix}$	$R^{-1}SR^{-1}SRSRSRSR^{-1}$	89	74
93	$\begin{pmatrix} 6 & 1 \\ -1 & 0 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}SR^{-1}$	85	77
94	$\begin{pmatrix} 9 & -2 \\ -4 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SRSRSRSR$	86	76
95	$\begin{pmatrix} 9 & -4 \\ -2 & 1 \end{pmatrix}$	$R^{-1}SR^{-1}SR^{-1}SR^{-1}SRSR$	64	75
96	$\begin{pmatrix} 30 & 17 \\ -23 & -13 \end{pmatrix}$	$R^{-1}SRSRSRSR^{-1}SR^{-1}SR^{-1}SRSR^{-1}$	87	73

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